NUMERICAL RADIUS NORMS ON OPERATOR SPACES

TAKASHI ITOH* AND MASARU NAGISA**

Abstract. We introduce a numerical radius operator space $(X, \mathcal{W}_n)$. The conditions to be a numerical radius operator space are weaker than the Ruan's axiom for an operator space $(X, \mathcal{O}_n)$. Let $w(\cdot)$ be the numerical radius norm on $\mathbb{B}(\mathcal{H})$. It is shown that if X admits a norm $\mathcal{W}_n(\cdot)$ on the matrix space $\mathbb{M}_n(X)$ which satisfies the conditions, then there is a complete isometry, in the sense of the norms $\mathcal{W}_n(\cdot)$ and $w_n(\cdot)$, from $(X, \mathcal{W}_n)$ into $(\mathbb{B}(\mathcal{H}), w_n)$. We study the relationship between the operator space $(X, \mathcal{O}_n)$ and the numerical radius operator space $(X, \mathcal{W}_n)$. The category of operator spaces can be regarded as a subcategory of numerical radius operator spaces.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ be the set of all bounded operators on a Hilbert space $\mathcal{H}$, and $\mathcal{H}^n$ the n-direct sum of $\mathcal{H}$. We denote by $\|a\|_n$ the operator norm, and $w_n(a)$ the numerical radius norm for $a \in \mathbb{B}(\mathcal{H}^n)$ respectively, and identify $\mathbb{B}(\mathcal{H}^n)$ with the $n \times n$ matrix space $\mathbb{M}_n(\mathbb{B}(\mathcal{H}))$.

In [11], Ruan introduced a striking concept of operator spaces. An (abstract) operator space is a complex linear space X together with a sequence of norms $\mathcal{O}_n(\cdot)$ on the $n \times n$ matrix space $\mathbb{M}_n(X)$ for each $n \in \mathbb{N}$, which satisfies the following Ruan's axioms OI, OII:

$$\text{OI.} \qquad \mathcal{O}_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\{\mathcal{O}_m(x), \mathcal{O}_n(y)\},$$

$$\text{OII.} \qquad \mathcal{O}_n(\alpha x \beta) \le \|\alpha\| \mathcal{O}_m(x) \|\beta\|$$

for all $x \in \mathbb{M}_m(X), y \in \mathbb{M}_n(X)$ and $\alpha \in M_{n,m}(\mathbb{C}), \beta \in M_{m,n}(\mathbb{C})$.

Ruan proved in [11] that if X is an (abstract) operator space, then there is a complete isometry Ψ from X to $\mathbb{B}(\mathcal{H})$, that is, $\|[\Psi(x_{ij})]\|_n = \mathcal{O}_n([x_{ij}])$ for all $[x_{ij}] \in \mathbb{M}_n(X), n \in \mathbb{N}$.

In this paper, we introduce an (abstract) numerical radius operator space. We call that X is a numerical radius operator space if a complex

linear space X admits a sequence of norms $\mathcal{W}_n(\cdot)$ on the $n \times n$ matrix space $\mathbb{M}_n(X)$ for each $n \in \mathbb{N}$, which satisfies a couple of conditions WI, WII, where WI is the same as OI, however WII is a slightly weaker condition than OII as follows:

$$\text{WI.} \qquad \mathcal{W}_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\{\mathcal{W}_m(x), \mathcal{W}_n(y)\},$$

$$\text{WII.} \qquad \mathcal{W}_n(\alpha x \alpha^*) \leq \|\alpha\|^2 \mathcal{W}_m(x),$$

for all $x \in \mathbb{M}_m(X), y \in \mathbb{M}_n(X)$ and $\alpha \in M_{n,m}(\mathbb{C})$.

It is clear that a subspace $X \subset \mathbb{B}(\mathcal{H})$ is a (concrete) numerical radius operator space with $w_n(\cdot)$.

We first show that if X is a numerical radius operator space, then there is a complete isometry Φ, in the sense of norms $w_n([\Phi(x_{ij})]) = \mathcal{W}_n([x_{ij}])$ for all $[x_{ij}] \in \mathbb{M}_n(X), n \in \mathbb{N}$, from X to a concrete numerical radius operator space in $\mathbb{B}(\mathcal{H})$.

It is well known that there is an equality between the operator norm and the numerical radius norm so that

$$\frac{1}{2}\|x\| = w\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \qquad \text{for } x \in \mathbb{B}(\mathcal{H}).$$

We next show that, given a numerical radius operator space X with $\mathcal{W}_n$, defining $\mathcal{O}_n$ by

$$\text{(OW)} \qquad \frac{1}{2}\mathcal{O}_n(x) = \mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \qquad \text{for } x \in \mathbb{M}_n(X),$$

X becomes an operator space with $\mathcal{O}_n$. On the other hand, given an operator space X with $\mathcal{O}_n$, the numerical radius operator space which satisfies the equality (OW) is not unique. More precisely, for every operator space X, there exists the maximal (resp. minimal) numerical radius norm $\mathcal{W}_{\max}$ (resp.$\mathcal{W}_{\min}$) affiliated with $(X, \mathcal{O}_n)$ (See the definition in section 3) among all of $\mathcal{W}$'s which satisfy WI, WII and (OW). Moreover it is shown that $\mathcal{W}_{\min} \leq \mathcal{W}_{\max} \leq 2\mathcal{W}_{\min}$, and there are uncountably many $\mathcal{W}$'s which satisfy WI, WII and (OW) such that

$$\mathcal{W}_{\min}(x) \leq \mathcal{W}(x) \leq \mathcal{W}_{\max}(x) \qquad \text{for all } x \in \mathbb{M}_n(X), n \in \mathbb{N}.$$

Let $\mathbb{O}$ be the category of the operator spaces in which the objects are operator spaces and the morphisms are completely bounded maps, $\mathbb{W}$ the category of the numerical radius operator spaces in which the objects are numerical radius operator spaces and the morphisms are $\mathcal{W}$-completely bounded maps (See the definition in section 2). We finally show that $\mathcal{W}_{\min}$ and $\mathcal{W}_{\max}$ are the strict functors which embed $\mathbb{O}$ into $\mathbb{W}$.

2. Numerical radius operator spaces

In this secton, we are going to prove a representation theorem for abstract numerical radius operator spaces.

Given abstract numerical radius operator spaces (or operator spaces) X, Y and a linear map φ from X to Y, φ_n from $\mathbb{M}_n(X)$ to $\mathbb{M}_n(Y)$ is defined to be

$$\varphi_n([x_{ij}]) = [\varphi(x_{ij})] \qquad \text{for each } [x_{ij}] \in \mathbb{M}_n(X),\ n \in \mathbb{N}.$$

We use a simple notation for the norm of $x = [x_{ij}] \in \mathbb{M}_n(X)$ to be $\mathcal{W}(x)$ (resp. $\mathcal{O}(x)$) instead of $\mathcal{W}_n(x)$ (resp. $\mathcal{O}_n(x)$), and for the norm of $f \in \mathbb{M}_n(X)^*$ to be $\mathcal{W}^*(f) = \sup\{|f(x)||x = [x_{ij}] \in \mathbb{M}_n(X), \mathcal{W}(x) \leq 1\}$. We denote the norm of φ_n by $\mathcal{W}(\varphi_n) = \sup\{\mathcal{W}(\varphi_n(x))|x = [x_{ij}] \in \mathbb{M}_n(X), \mathcal{W}(x) \leq 1\}$ (resp. $\mathcal{O}(\varphi_n) = \sup\{\mathcal{O}(\varphi_n(x))|x = [x_{ij}] \in \mathbb{M}_n(X), \mathcal{O}(x) \leq 1\}$. The $\mathcal{W}$-completely bounded norm (resp. completely bounded norm) of φ is defined to be

$$\mathcal{W}(\varphi)_{cb} = \sup\{\mathcal{W}(\varphi_n)|n \in \mathbb{N}\},$$

$$(\text{resp.} \quad \mathcal{O}(\varphi)_{cb} = \sup\{\mathcal{O}(\varphi_n)|n \in \mathbb{N}\}).$$

We say φ is $\mathcal{W}$-completely bounded (resp. completely bounded) if $\mathcal{W}(\varphi)_{cb} < \infty$ (resp.$\mathcal{O}(\varphi)_{cb} < \infty$), and φ is $\mathcal{W}$-completely contractive (resp. completely contractive) if $\mathcal{W}(\varphi)_{cb} \leq 1$ (resp.$\mathcal{O}(\varphi)_{cb} \leq 1$). We call φ is a $\mathcal{W}$-complete isometry (resp. complete isometry) if $\mathcal{W}(\varphi_n(x)) = \mathcal{W}(x)$ (resp.$\mathcal{O}(\varphi_n(x)) = \mathcal{O}(x)$) for each $x \in \mathbb{M}_n(X)$, $n \in \mathbb{N}$.

The next is fundamental in numerical radius operator spaces like the Ruan's Theorem [11] in the operator space theory.

Theorem 2.1. *If X is a numerical radius operator space with $\mathcal{W}_n$, then there exist a Hilbert space $\mathcal{H}$, a concrete numerical radius operator space $Y \subset \mathbb{B}(\mathcal{H})$ with the numerical radius norm $w(\cdot)$, and a $\mathcal{W}$-complete isometry Φ from $(X, \mathcal{W}_n)$ onto (Y, w_n).*

To prove this theorem, we use the similar argument and idea as in the proof of [3]. We just follow each step of the proof in [3], however we write it down for the convenience of the reader because Theorem 2.1 also implies the Ruan's Theorem (See Corollary 2.5). The conditions WI and WII work in the next Lemma.

Lemma 2.2. *Let X be a numerical radius operator space. If $f \in \mathbb{M}_n(X)^*$ and $\mathcal{W}^*(f) \leq 1$, then there exists a state p_0 on $\mathbb{M}_n(\mathbb{C})$ such*

that

$$
\begin{array}{lrl}
(1) & |f(\alpha x\alpha^*)| \leq & p_0(\alpha\alpha^*)\mathcal{W}(x), \\
(2) & |f(\alpha x\beta)| \leq & 2p_0(\alpha\alpha^*)^{\frac{1}{2}}p_0(\beta^*\beta)^{\frac{1}{2}}\mathcal{W}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)
\end{array}
$$

for all $\alpha \in \mathbb{M}_{n,r}(\mathbb{C}), x \in \mathbb{M}_r(X), \beta \in \mathbb{M}_{r,n}(\mathbb{C}), r \in \mathbb{N}$.

Proof. First, we prove the inequality (1). It is sufficient to show the existence of a state p_0 in the state space $S(\mathbb{M}_n(\mathbb{C}))$ of $\mathbb{M}_n(\mathbb{C})$ such that

$$\mathrm{Re}f(\alpha x\alpha^*) \leq p_0(\alpha\alpha^*)\mathcal{W}(x) \qquad \text{for all } \alpha \in \mathbb{M}_{n,r}(\mathbb{C}), x \in \mathbb{M}_r(X).$$

For $\alpha_i \in \mathbb{M}_{n,r_i}(\mathbb{C})$ and $x_i \in \mathbb{M}_{r_i}(V)$ with $\mathcal{W}(x_i) \leq 1$ $(i = 1, \ldots, k,\ k \in \mathbb{N})$, we define a real valued finction $F_{\{\alpha_1,\ldots,\alpha_k,x_1,\ldots,x_k\}}(\ \cdot\ )$ on $S(\mathbb{M}_n(\mathbb{C}))$ by

$$F_{\{\alpha_1,\ldots,\alpha_k,x_1,\ldots,x_k\}}(p) = \sum_{i=1}^{k} p(\alpha_i\alpha_i^*) - \mathrm{Re}f(\alpha_i x_i\alpha_i^*), \text{ for } p \in S(\mathbb{M}_n(\mathbb{C})).$$

Set

$$\triangle = \{F_{\{\alpha_1,\ldots,\alpha_k,x_1,\ldots,x_k\}} \mid \alpha_i \in \mathbb{M}_{n,r_i}(\mathbb{C}), x_i \in \mathbb{M}_{r_i}(V), \mathcal{W}(x_i) \leq 1, r_i, k \in \mathbb{N}\}.$$

It is easy to see that $\triangle$ is a cone in the set of all real functions on $S(\mathbb{M}_n(\mathbb{C}))$. Let $\bigtriangledown$ be the open cone of all strictly negative functions on $S(\mathbb{M}_n(\mathbb{C}))$. For any $\alpha_i \in \mathbb{M}_{n,r_i}(\mathbb{C}), i = 1, \cdots, k$, there exists $p_1 \in S(\mathbb{M}_n(\mathbb{C}))$ such that $p_1(\sum\alpha_i\alpha_i^*) = \|\sum\alpha_i\alpha_i^*\|$.

Since

$$
\begin{aligned}
\mathcal{W}(\sum\alpha_i x_i\alpha_i^*) &= \mathcal{W}\left([\alpha_1, \cdots, \alpha_k]\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{bmatrix}[\alpha_1, \cdots, \alpha_k]^*\right) \\
&\leq \|[\alpha_1, \cdots, \alpha_k]\|^2\mathcal{W}\left(\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{bmatrix}\right) \\
&= \|\sum\alpha_i\alpha_i^*\|\max_i\{\mathcal{W}(x_i)\} \\
&\leq \|\sum\alpha_i\alpha_i^*\|,
\end{aligned}
$$

we have

$$\begin{aligned}
F_{\{\alpha_1,\ldots,\alpha_k,x_1,\ldots,x_k\}}(p_1) &= \sum p_1(\alpha_i\alpha_i^*) - \mathrm{Re}\sum f(\alpha_i x_i \alpha_i^*) \\
&\geq \|\sum \alpha_i\alpha_i^*\| - |f(\sum \alpha_i x_i \alpha_i^*)| \\
&\geq \|\sum \alpha_i\alpha_i^*\| - \mathcal{W}(\sum \alpha_i x_i \alpha_i^*) \\
&\geq 0.
\end{aligned}$$

Thus it turns out $\triangle \cap \bigtriangledown = \emptyset$.

By the Hahn-Banach Theorem, there exists a measure μ on $S(\mathbb{M}_n(\mathbb{C}))$ such that $\mu(\triangle) \geq 0$ and $\mu(\bigtriangledown) < 0$. So we may assume that μ is a probability measure. Set $p_0 = \int p d\mu(p)$. Since $F_{\{\alpha, \frac{x}{\mathcal{W}(x)}\}} \in \triangle$, we obtain

$$p_0(\alpha\alpha^*) - \mathrm{Re} f(\alpha\frac{x}{\mathcal{W}(x)}\alpha^*) = \int F_{\{\alpha,\frac{x}{\mathcal{W}(x)}\}}(p)d\mu(p) \geq 0.$$

Next, we prove the inequality (2). Since

$$|f(\alpha x\beta)| = \left| f\left([\alpha, \beta^*] \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} [\alpha, \beta^*]^* \right) \right|,$$

we have

$$|f(\alpha x\beta)| \leq p_0(\alpha\alpha^* + \beta^*\beta)\mathcal{W}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right).$$

Let $\lambda > 0$ and replace α, β by $\lambda\alpha, \lambda^{-1}\beta$. Then the equality

$$\inf_{\lambda>0}\{\lambda^2 p_0(\alpha\alpha^*) + \lambda^{-2}p_0(\beta^*\beta)\} = 2p_0(\alpha\alpha^*)^{\frac{1}{2}}p_0(\beta^*\beta)^{\frac{1}{2}}$$

implies the desired inequality (2).

□

The next is known as Smith's Lemma [12] in case that X is an operator space.

Lemma 2.3. *Let X be a numerical radius operator space. If φ is a linear map from X to $\mathbb{M}_n(\mathbb{C})$, then*

$$\mathcal{W}(\varphi)_{cb} = \mathcal{W}(\varphi_n).$$

Proof. We can follow the same argument as in the proof of Smith's. □

Lemma 2.4. *Let X be a numerical radius operator space. If $f \in \mathbb{M}_n(X)^*$ and $\mathcal{W}^*(f) \leq 1$, then there exist a $\mathcal{W}$-complete contraction from X to $\mathbb{M}_n(\mathbb{C})$ and a unit vector $\xi \in (\mathbb{C}^n)^n$ such that*

$$f(x) = (\varphi_n(x)\xi|\xi) \qquad \textit{for all } x \in \mathbb{M}_n(X).$$

Proof. Let p_0 be a state which satisfies the inequalities in Lemma 2.2. By the GNS construction for p_0, we have a representation π of $\mathbb{M}_n(\mathbb{C})$ on a finite dimensional Hilbert space $\mathcal{H}$ and a cyclic vector $\xi_0 \in \mathcal{H}$ such that

$$p_0(\alpha) = (\pi(\alpha)\xi_0|\xi_0).$$

For $\alpha = [\alpha_1, \dots, \alpha_n] \in \mathbb{M}_{1,n}(\mathbb{C})$, we set $\tilde{\alpha} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \\ & 0 & \end{bmatrix} \in \mathbb{M}_n(\mathbb{C})$ and denote by $\tilde{\mathbb{M}}_n(\mathbb{C})$ all of the elements in the form $\tilde{\alpha}$. Let $\mathcal{H}_0 = \pi(\tilde{\mathbb{M}}_n(\mathbb{C}))\xi_0$. For a fixed $x \in X$, define a quasilinear form B_x on $\mathcal{H}_0 \times \mathcal{H}_0$ by

$$B_x(\pi(\tilde{\beta})\xi_0, \pi(\tilde{\alpha})\xi_0) = f(\alpha^* x \beta).$$

Since

$$\begin{aligned} |f(\alpha^* x\beta)| &\leq p_0(\alpha^*\alpha)^{\frac{1}{2}} p_0(\beta^*\beta)^{\frac{1}{2}} 2\mathcal{W}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \\ &= \|\pi(\tilde{\alpha})\xi_0\| \|\pi(\tilde{\beta})\xi_0\| 2\mathcal{W}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right), \end{aligned}$$

$B_x(\cdot,\cdot)$ is well-defined, and there exists a bounded operator $\varphi_0(x) \in \mathbb{B}(\mathcal{H}_0)$ such that

$$f(\alpha^* x\beta) = (\varphi_0(x)\pi(\tilde{\beta})\xi_0|\pi(\tilde{\alpha})\xi_0).$$

Since $\dim \mathcal{H}_0 \leq n$, we may assume that $\mathcal{H}_0$ is a subspace of $\mathbb{C}^n$. Let e be a projection from $\mathbb{C}^n$ onto $\mathcal{H}_0$. Set $\varphi(x) = \varphi_0(x)e$ for $x \in X$. Then it turns out that φ maps from X to $\mathbb{M}_n(\mathbb{C})$ and

$$f(\alpha^* x\beta) = (\varphi(x)\pi(\tilde{\beta})\xi_0|\pi(\tilde{\alpha})\xi_0).$$

We let $e_j = [0, \dots, \overset{(j)}{1}, \dots, 0] \in \mathbb{M}_{1,n}(\mathbb{C})$ and $\xi = \begin{bmatrix} \pi(\tilde{e}_1)\xi_0 \\ \vdots \\ \pi(\tilde{e}_n)\xi_0 \end{bmatrix}$. Then it is not hard to see that

$$f(x) = (\varphi_n(x)\xi|\xi) \quad \text{for } x \in \mathbb{M}_n(X) \quad \|\xi\| \leq 1.$$

To prove the $\mathcal{W}$-complete boundedness of φ, by Lemma 2.3, we let $x = [x_{ij}] \in \mathbb{M}_n(X)$ and $\xi_1 = \begin{bmatrix} \pi(\tilde{\gamma}_1)\xi_0 \\ \vdots \\ \pi(\tilde{\gamma}_n)\xi_0 \end{bmatrix}$ with $\|\xi_1\| \leq 1$ where $\gamma_i \in$

$\mathbb{M}_{1,n}(\mathbb{C})$. Set $\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} \in \mathbb{M}_n(\mathbb{C})$. Then it turns out that

$$p_0(\gamma^*\gamma) = \sum_i \|\pi(\tilde{\gamma_i})\xi_0\|^2 = \|\xi_1\|^2 \leq 1.$$

Thus we have

$$\begin{aligned} |(\varphi_n(x)\xi_1|\xi_1)| &= |\sum(\varphi_0(x_{ij})\pi(\tilde{\gamma_j})\xi_0|\pi(\tilde{\gamma_j})\xi_0)| \\ &= |\sum f(\gamma_i^* x_{ij}\gamma_j)| \\ &= |f(\gamma^* x\gamma)| \\ &\leq p_0(\gamma^*\gamma)\mathcal{W}(x) \\ &= \mathcal{W}(x). \end{aligned}$$

□

Now we will prove the Theorem 2.1. We denote by $\mathcal{W}CB(X,Y)$ the set of all $\mathcal{W}$-completely bounded maps from X to a numerical radius operator space Y.

Proof of Theorem 2.1

Let $\mathcal{C} = \cup_{n\in\mathbb{N}}\{\varphi \in \mathcal{W}CB(X, \mathbb{M}_n(\mathbb{C})) \mid \mathcal{W}(\varphi)_{cb} \leq 1\}$ and $\mathcal{H} = \oplus_{\varphi\in\mathcal{C}}\mathbb{C}^{n(\varphi)}$, where $n(\varphi)$ is the degree of the range space $\mathbb{M}_{n(\varphi)}(\mathbb{C})$ of φ. Define that

$$\Phi : X \ni x \longmapsto (\varphi(x))_\varphi \in \oplus_{\varphi\in\mathcal{C}}\mathbb{M}_{n(\varphi)}(\mathbb{C}).$$

Since $\mathcal{W}(\varphi)_{cb} \leq 1$, it is clear that $\mathcal{W}(\Phi)_{cb} \leq 1$. Conversely, given any $x \in \mathbb{M}_n(X)$, there exists $f \in \mathbb{M}_n(X)^*$ with $\mathcal{W}^*(f) \leq 1$ such that $f(x) = \mathcal{W}(x)$ by the Hahn-Banach Theorem. By Lemma 2.4, we find $\varphi \in \mathcal{W}CB(X, \mathbb{M}_n(\mathbb{C}))$ with $\mathcal{W}(\varphi)_{cb} \leq 1$ and a unit vector $\xi \in (\mathbb{C}^n)^n$ such that $f(x) = (\varphi_n(x)\xi|\xi)$. Thus it turns out $w(\varphi_n(x)) = \mathcal{W}(x)$. Hence we obtain that $w(\Phi_n(x)) \geq w(\varphi_n(x)) = \mathcal{W}(x)$. This completes the proof.

Corollary 2.5. (Ruan's Theorem [11]) If X is an operator space with $\mathcal{O}_n$, then there exist a Hilbert space $\mathcal{H}$, a concrete operator space $Y \subset \mathbb{B}(\mathcal{H})$, and a complete isometry Ψ from $(X, \mathcal{O}_n)$ onto $(Y, \| \ \|_n)$.

Proof. Since $(X, \mathcal{O}_n)$ is also a numerical radius operator space, we can find a W-complete isometry Φ from $(X, \mathcal{O}_n)$ into $(B(H), w_n)$ by Theorem 2.1. We put $\Psi(x) = \frac{1}{2}\Phi(x)$. Then we have for $x \in \mathbb{M}_n(X)$,

$$\begin{aligned}
&\|\Psi_n(x)\|_n \leq 2w_n(\Psi_n(x)) = w_n(\Phi_n(x)) \\
=&\mathcal{O}_n(x) = \mathcal{O}_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{O}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\
\leq&\mathcal{O}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = w_{2n}\left(\begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix}\right) = 2w_{2n}\left(\begin{bmatrix} 0 & \Psi_n(x) \\ 0 & 0 \end{bmatrix}\right) \\
=&\|\Psi_n(x)\|_n.
\end{aligned}$$

□

Corollary 2.6. *If X is a numerical radius operator space with $\mathcal{W}_n$, then there exist an operator space norm $\mathcal{O}_n$ on X and a complete & $\mathcal{W}$-complete isometry Φ from X into $\mathbb{B}(\mathcal{H})$.*

Proof. For given $\mathcal{W}_n$ and $x \in \mathbb{M}_n(X)$, we define $\mathcal{O}_n$ to be $\mathcal{O}_n(x) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)$. By Theorem 2.1, there exist a $\mathcal{W}$-complete isometry Φ from $(X, \mathcal{W}_n)$ into $(\mathbb{B}(\mathcal{H}), w_n)$. Since

$$\|\Phi_n(x)\|_n = 2w_{2n}\left(\begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix}\right) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \mathcal{O}_n(x),$$

Φ is also a complete isometry from $(X, \mathcal{O}_n)$ into $(\mathbb{B}(\mathcal{H}), \|\ \|_n)$.

□

As in the case of the operator space theory, we can see the basic operations are closed in numerical radius operator spaces X, Y. For $\varphi = [\varphi_{ij}] \in \mathbb{M}_n(\mathcal{W}CB(X, Y))$, we use the identification $\mathbb{M}_n(\mathcal{W}CB(X, Y)) = \mathcal{W}CB(X, \mathbb{M}_n(Y))$ by $\varphi(x) = [\varphi_{ij}(x)]$ for $x \in X$ with the norm $\mathcal{W}(\varphi)_{cb}$. Especially, $\mathbb{M}_n(X^*)$ is identified with $\mathcal{W}CB(X, \mathbb{M}_n(\mathbb{C}))$ where we give the numerical radius norm $w(\cdot)$ on $\mathbb{M}_n(\mathbb{C})$.

If N is a closed subspace of X, we use the identification $\mathbb{M}_n(X/N) = \mathbb{M}_n(X)/\mathbb{M}_n(N)$.

Here we state only the fundamental operations.

Proposition 2.7. *Suppose that X and Y are numerical radius operator spaces. Then*

(1) *$\mathcal{W}CB(X, Y)$ is a numerical radius operator space.*
(2) *The canonical inclusion $X \hookrightarrow X^{**}$ is $\mathcal{W}$-completely isometric.*
(3) *If N is a closed subspace of X, then X/N is a numerical radius operator space.*

Proof. For (1) and (3), it is not hard to verify that the norms defined as above on $\mathbb{M}_n(\mathcal{W}CB(X,Y))$ and $\mathbb{M}_n(X/N)$ satisfy the conditions WI and WII.

To show (2), since the inclusion $i : \mathbb{M}_n(X) \ni x \longmapsto i(x) \in \mathbb{M}_n(X^{**})$ is defined by

$$< i(x), f >=< f, x >= w([f_{ij}(x_{kl})]) \qquad \text{for } x \in \mathbb{M}_n(X),\ f \in \mathbb{M}_n(X^*),$$

we have

$$\mathcal{W}(i(x))_{cb} = \sup\{| < f, x > | \ \ | f \in \mathbb{M}_n(X^*),\ \mathcal{W}_{cb}(f) \le 1\} = \mathcal{W}(x)$$

by Lemma 2.4. □

3. Numerical radius norms and operator spaces

In this section, we study the relationship between numerical radius operator spaces and operator spaces.

Let X be a numerical radius operator space with $\mathcal{W}_n$. Defining by $\mathcal{O}_n(x) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)$ for $x \in \mathbb{M}_n(X)$, $(X, \mathcal{O}_n)$ is an operator space from Corollary 2.6.

On the other hand, we let X be an operator space with $\mathcal{O}_n$. We call that a sequence of norms $\mathcal{W}_n$ is a numerical radius norm affiliated with $(X, \mathcal{O}_n)$ if $\mathcal{W}_n$ satisfies WI, WII and

$$\text{(OW)} \qquad \frac{1}{2}\mathcal{O}_n(x) = \mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \qquad \text{for } x \in \mathbb{M}_n(X).$$

We often write $\mathcal{W}$(resp. $\mathcal{O}$) instead of $\mathcal{W}_n$ (resp. $\mathcal{O}_n$).

Definition 3.1. We define a norm $\mathcal{W}_{\max}$ on an operator space $(X, \mathcal{O}_n)$ by

$$\mathcal{W}_{\max}(x) = \inf \frac{1}{2}\|aa^* + b^*b\| \quad \text{for } x \in \mathbb{M}_n(X),$$

where the infimum is taken over all $a \in \mathbb{M}_{n,r}(\mathbb{C}), y \in \mathbb{M}_r(X), b \in \mathbb{M}_{r,n}(\mathbb{C}), r \in \mathbb{N}$ such that $x = ayb$ and $\mathcal{O}(y) = 1$.

We call $\mathcal{W}_{\max}$ is the maximal numerical radius norm affiliated with $(X, \mathcal{O}_n)$. We note that a, y, b can be chosen from $a \in \mathbb{M}_n(\mathbb{C}), y \in \mathbb{M}_n(X), b \in \mathbb{M}_n(\mathbb{C}), n \in \mathbb{N}$ in the definition of $\mathcal{W}_{\max}$ by using the right polar decomposition of $a = |a^*|u$ and the left polar decomposition of $b = v|b|$.

It is easy to see that, for $x \in \mathbb{M}_n(X)$, we have

$$\mathcal{O}(x) = \inf \|a\|\|b\|$$

where the infimum is taken over all $x = ayb$ as in Definition 3.1. Then it follows that

$$\frac{1}{2}\mathcal{O}(x) \le \mathcal{W}_{\max}(x) \le \mathcal{O}(x) \quad \text{for } x \in \mathbb{M}_n(X).$$

Theorem 3.2. *Suppose that X is an operator space with $\mathcal{O}_n$. Then $\mathcal{W}_{max}$ is a numerical radius norm affiliated with $(X, \mathcal{O}_n)$ and the maximal among all of numerical radius norms affiliated with $(X, \mathcal{O}_n)$.*

Proof. First we show that $\mathcal{W}_{\max}$ is a norm. To see that $\mathcal{W}_{\max}(x_1+x_2) \le \mathcal{W}_{\max}(x_1)+\mathcal{W}_{\max}(x_2)$ for $x_1, x_2 \in \mathbb{M}_n(X)$, let $x_i = a_i y_i b_i, \mathcal{O}(y_i) = 1 (i = 1, 2)$. Since

$$x_1 + x_2 = [a_1, a_2] \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ and } \mathcal{O}\left(\begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}\right) = 1,$$

we have

$$\begin{aligned} \mathcal{W}_{\max}(x_1 + x_2) &\le \frac{1}{2}\|a_1 a_1^* + a_2 a_2^* + b_1^* b_1 + b_2^* b_2\| \\ &\le \frac{1}{2}\|a_1 a_1^* + b_1^* b_1\| + \frac{1}{2}\|a_2 a_2^* + b_2^* b_2\|. \end{aligned}$$

It is easy to show the rest of the norm conditions.

Next we prove that $\mathcal{W}_{\max}$ satisfies WI and WII. To see WI, let $\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} = ayb$ and $\mathcal{O}(y) = 1$. Since $x_1 = [1, 0]ayb \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have

$$\begin{aligned} \mathcal{W}_{\max}(x_1) &\le \frac{1}{2}\|[1, 0]aa^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [1, 0]b^* b \begin{bmatrix} 1 \\ 0 \end{bmatrix}\| \\ &\le \frac{1}{2}\|aa^* + b^* b\|. \end{aligned}$$

Also we have $\mathcal{W}_{\max}(x_2) \le \frac{1}{2}\|aa^* + b^* b\|$. Thus it turns out that

$$\mathcal{W}_{\max}\left(\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right) \le \max\{\mathcal{W}_{\max}(x_1), \mathcal{W}_{\max}(x_2)\}.$$

Conversely, let $x_i = a_i y_i b_i, \mathcal{O}(y_i) = 1$ $(i = 1, 2)$. Since

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix},$$

we have

$$\mathcal{W}_{\max}\left(\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\begin{bmatrix} a_1a_1^* + b_1^*b_1 & 0 \\ 0 & a_2a_2^* + b_2^*b_2 \end{bmatrix}\right\|$$
$$\leq \max\{\frac{1}{2}\|a_1a_1^* + b_1^*b_1\|, \frac{1}{2}\|a_2a_2^* + b_2^*b_2\|\}.$$

To see WII, let $x = ayb, \mathcal{O}(y) = 1$ and $\alpha \in \mathbb{M}_n(\mathbb{C})$. Then

$$\begin{aligned} \mathcal{W}_{\max}(\alpha x \alpha^*) &\leq \frac{1}{2}\|\alpha a a^* \alpha^* + \alpha b^* b \alpha^*\| \\ &= \frac{1}{2}\|\alpha(aa^* + b^*b)\alpha^*\| \\ &\leq \frac{1}{2}\|aa^* + b^*b\|\|\alpha\alpha^*\|. \end{aligned}$$

To see the condition (OW), let $\mathcal{O}(x) = 1$. Since

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and } \mathcal{O}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = 1,$$

we have

$$\mathcal{W}_{\max}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^* + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^*\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\| = \frac{1}{2}.$$

To get the other inequality, let $x \in \mathbb{M}_r(X)$ with $\mathcal{O}(x) = 1$. By the Ruan's Theorem, there exist a complete isometry $\varphi : X \longrightarrow \mathbb{B}(\mathcal{H})$. Given $\varepsilon > 0$, we find a unit vectors $\xi, \eta \in \mathcal{H}^r$ such that $1 - \varepsilon < (\varphi_r(x)\xi|\eta)$. Define $F \in \mathbb{M}_{2r}(X)^*$ for $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in \mathbb{M}_{2r}(X)$ by

$$F\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \left(\begin{bmatrix} \varphi_r(x_1) & \varphi_r(x_2) \\ \varphi_r(x_3) & \varphi_r(x_4) \end{bmatrix}\begin{bmatrix} 0 \\ \xi \end{bmatrix} \,\middle|\, \begin{bmatrix} \eta \\ 0 \end{bmatrix}\right).$$

We show that $\mathcal{W}_{\max}^*(F) \leq 2$. Let $z \in \mathbb{M}_{2r}(X)$ with $\mathcal{W}_{\max}(z) < 1$. We may assume that $z = ayb, \mathcal{O}(y) = 1$ and $\|aa^* + b^*b\| < 2$ where

$y \in \mathbb{M}_k(X), a \in \mathbb{M}_{2r,k}(\mathbb{C})$ and $b \in \mathbb{M}_{k,2r}(\mathbb{C})$. Since

$$\begin{aligned} F(z) &= (a\varphi_k(y)b \begin{bmatrix} 0 \\ \xi \end{bmatrix} | \begin{bmatrix} \eta \\ 0 \end{bmatrix}) \\ &= \left(\begin{bmatrix} 0 & \varphi_k(y) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^* \\ b \end{bmatrix} \begin{bmatrix} 0 \\ \xi \end{bmatrix} | \begin{bmatrix} a^* \\ b \end{bmatrix} \begin{bmatrix} \eta \\ 0 \end{bmatrix} \right) \\ &\leq \|\varphi_k(y)\| \left\| \begin{bmatrix} a^* \\ b \end{bmatrix} \right\|^2 < 2, \end{aligned}$$

we obtain that

$$\begin{aligned} \mathcal{W}_{\max}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) &\geq \frac{1}{2} F\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \left| \left(\begin{bmatrix} 0 & \varphi_r(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \xi \end{bmatrix} \mid \begin{bmatrix} \eta \\ 0 \end{bmatrix} \right) \right| \\ &= \frac{1}{2} |(\varphi_r(x)\xi|\eta)| \\ &> \frac{1-\varepsilon}{2} \end{aligned}$$

Finally we show the maximality of $\mathcal{W}_{\max}$ in the set of all numerical radius norms affiliated with $(X, \mathcal{O}_n)$. To see this, let $\mathcal{W}$ be an arbitrary numerical radius norm affiliated with $(X, \mathcal{O}_n)$ and $x = ayb, y \in \mathbb{M}_k(X), a \in \mathbb{M}_{n,k}(\mathbb{C})$ and $b \in \mathbb{M}_{k,n}(\mathbb{C})$. Then we have

$$\begin{aligned} \mathcal{W}(x) &= \mathcal{W}\left([a, b^*] \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^* \\ b \end{bmatrix} \right) \\ &\leq \|[a, b^*]\|^2 \mathcal{W}\left(\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \|aa^* + b^*b\| \mathcal{O}(y) \end{aligned}$$

This implies that $\mathcal{W}(x) \leq \mathcal{W}_{\max}(x)$ and completes the proof. $\square$

Next we set $\mathcal{W}_{\min}(x) = \frac{1}{2}\mathcal{O}(x)$ for $x \in \mathbb{M}_n(X)$. It is clear that $\mathcal{W}_{\min}$ satisfies WI, WII and (OW). We can characterize numerical radius norms affiliated with an operator space X by using $\mathcal{W}_{\min}$ and $\mathcal{W}_{\max}$. We call $\mathcal{W}_{\min}$ is the minimal numerical radius norm affiliated with $(X, \mathcal{O}_n)$.

Corollary 3.3. *Suppose that X is an operator space with $\mathcal{O}_n$, and $\mathcal{W}_n$ satisfies WI, WII. Then the following are equivalent:*

(1) (OW) $\frac{1}{2}\mathcal{O}_n(x) = \mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)$ for $x \in \mathbb{M}_n(X)$,

(2) There exists a complete and $\mathcal{W}$-complete isometry $\Phi : X \longrightarrow \mathbb{B}(\mathcal{H})$,

(3) $\mathcal{W}_{\min}(x) \le \mathcal{W}(x) \le \mathcal{W}_{\max}(x)$ for $x \in \mathbb{M}_n(X)$.

Proof. (1) $\Rightarrow$ (2) It follows from the same argument as in the proof of Corollary 2.6.

(2) $\Rightarrow$ (3) Let $x \in \mathbb{M}_n(X)$. Then we have $\mathcal{W}_{\min}(x) = \frac{1}{2}\mathcal{O}(x) = \frac{1}{2}\|\Phi_n(x)\| \le w(\Phi_n(x)) = \mathcal{W}(x)$ and $\mathcal{W}(x) \le \mathcal{W}_{\max}(x)$ by Theorem 3.2.

(3) $\Rightarrow$ (1) Let $x \in \mathbb{M}_n(X)$. Then we have

$$\frac{1}{2}\mathcal{O}(x) = \mathcal{W}_{\min}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \le \mathcal{W}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \le \mathcal{W}_{\max}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \frac{1}{2}\mathcal{O}(x).$$

$\square$

Example 3.4. Let $(X, \mathcal{O}_n)$ be an operator space. We present that there are uncountably many numerical radius norms affiliated with $(X, \mathcal{O}_n)$.

From Corollary 3.3, there exists a complete and $\mathcal{W}$-complete isometry $\Phi_{\max} : X \longrightarrow \mathbb{B}(\mathcal{H})$ when we introduce the maximal numerical radius norm $\mathcal{W}_{\max}$ on X. Let $0 \le t \le 1$ and

$$a_t = \begin{bmatrix} 0 & 1 & & & \\ & 0 & t & & \\ & & \ddots & \ddots & \\ & & & \ddots & t \\ & & & & 0 \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}), \quad n \ge 3.$$

Define that $\Phi_t(x) = \Phi_{\max}(x) \otimes a_t$ for $x \in X$. Since $\|a_t\| = 1$, then $\Phi_t : X \longrightarrow \mathbb{B}(\mathcal{H}) \otimes \mathbb{M}_n(\mathbb{C})$ is completely isometric. Set $\mathcal{W}_t(x) = w_m([\Phi_t(x_{ij})])$ for $x = [x_{ij}] \in \mathbb{M}_m(X)$. It is clear that $\mathcal{W}_t$ is a numerical radius norm affiliated with $(X, \mathcal{O}_n)$. We show that

$$\mathcal{W}_{\max}(x) \cos\frac{\pi}{n+1} \le \mathcal{W}_1(x) \le \mathcal{W}_{\max}(x) \quad \text{for } x \in \mathbb{M}_m(X),\ m \in \mathbb{N}.$$

To see this, given $x = [x_{ij}] \in \mathbb{M}_m(X)$ and $\varepsilon > 0$. Then there exists a unit vector $\xi \in \mathcal{H}^m$ such that $|([\Phi_{\max}(x_{ij})]\xi \mid \xi)| > w([\Phi_{\max}(x_{ij})]) - \varepsilon$. From [5], we can find a unit vector $\eta \in \mathbb{C}^n$ such that $w(a_1) =$

$|(a_1\eta \mid \eta)| = \cos\frac{\pi}{n+1}$. Then we obtain that

$$\begin{aligned}\mathcal{W}_1(x) &\geq |((([\Phi_{\max}(x_{ij})] \otimes a_1)\xi \otimes \eta \mid \xi \otimes \eta)| \\ &= |([\Phi_{\max}(x_{ij})]\xi \mid \xi)||(a_1\eta \mid \eta)| \\ &> (\mathcal{W}_{\max}(x) - \varepsilon)\cos\frac{\pi}{n+1}.\end{aligned}$$

This implies that $\mathcal{W}_{\max}(x)\cos\frac{\pi}{n+1} \leq \mathcal{W}_1(x)$. The second inequality is clear because of the maximality of $\mathcal{W}_{\max}$. We note that $\mathcal{W}_0 = \mathcal{W}_{\min}$. Since $[0,1] \ni t \longmapsto \mathcal{W}_t(x) \in \mathbb{C}$ is continuous, then there exist uncountably many distinct numerical radius norms $\{\mathcal{W}_t\}_{0\leq t\leq 1}$ affiliated with $(X, \mathcal{O}_n)$.

There are many ways to construct the numerical radius norms like $\{\mathcal{W}_t\}_{0\leq t\leq 1}$ affiliated with $(X, \mathcal{O}_n)$. For instance, replace a_t by

$$b_t = \begin{bmatrix} 0 & \sqrt{1-t} \\ 0 & \sqrt{t} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

Example 3.5. Let $\mathbb{C}1$ be the one dimensinal operator space. Then for $\alpha = [\alpha_{ij}] \in \mathbb{M}_n(\mathbb{C}1)$, we have

$$\mathcal{W}_{\max}(\alpha) = w(\alpha).$$

To see this, since $\mathcal{W}_{\max}(\alpha) = w([\alpha_{ij}z])$ for some $z \in \mathbb{B}(\mathcal{K})$ with $\|z\| = 1$, and α double commutes with $\begin{bmatrix} z & & \\ & \ddots & \\ & & z \end{bmatrix}$, we have $\mathcal{W}_{\max}(\alpha) \leq w(\alpha)$. This and the maximality of $\mathcal{W}_{\max}$ imply that

$$w(\alpha) = \inf\{\frac{1}{2}\|\beta\beta^* + \gamma^*\gamma\| \mid \alpha = \beta y\gamma, \|y\| = 1,\ \beta, y, \gamma \in \mathbb{M}_n(\mathbb{C})\}.$$

We note that the above equality for $w(\alpha)$ gives a simple proof of the Ando's Theorem in [1], in case $\dim \mathcal{H} < \infty$.

Example 3.6. Let X, Y be operator spaces in $\mathbb{B}(\mathcal{H})$. For $x \in \mathbb{M}_{n,r}(X)$ and $y \in \mathbb{M}_{r,n}(Y)$, we denote by $x \odot y$ the element $[\sum_{k=1}^r x_{ik} \otimes y_{kj}] \in \mathbb{M}_n(X \otimes Y)$. We note that each element $u \in \mathbb{M}_n(X \otimes Y)$ has a form $x \odot y$ for some $x \in \mathbb{M}_{n,r}(X)$, $y \in \mathbb{M}_{r,n}(Y)$ and $r \in \mathbb{N}$.

(a)

We define

$$\|u\|_{wh} = \inf\{\frac{1}{2}\|xx^* + y^*y\| \ \mid\ u = x \odot y \in \mathbb{M}_n(X \otimes Y)\}$$

for $u \in \mathbb{M}_n(X \otimes Y)$ (c.f. [7]). Then it is not hard to verify that $\| \ \|_{wh}$ satisfies the conditions WI and WII. Moreover $\| \ \|_{wh}$ is a numerical

radius norm affiliated with $(X\otimes_h Y, \|\ \|_h)$, where $X\otimes_h Y$ is the Haagerup tensor product operator space with the Haagerup norm $\|\ \|_h$, i.e.

$$\|u\|_h = \inf\{\|x\|\|y\| \ \mid \ u = x\odot y \in \mathbb{M}_n(X\otimes Y)\}.$$

To see (OW), given $u = x\odot y\in\mathbb{M}_n(X\otimes Y)$, we may assume that $\|x\| = \|y\|$. Since

$$\begin{aligned}
2\left\|\begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix}\right\|_{wh} &= 2\left\|\begin{bmatrix} x & 0 \\ 0 & 0\end{bmatrix}\odot\begin{bmatrix} 0 & y \\ 0 & 0\end{bmatrix}\right\|_{wh} \\
&\le \left\|\begin{bmatrix} x & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0\end{bmatrix}^* + \begin{bmatrix} 0 & y \\ 0 & 0\end{bmatrix}^*\begin{bmatrix} 0 & y \\ 0 & 0\end{bmatrix}\right\| \\
&\le \max\{\|x\|^2, \|y\|^2\} \\
&= \|x\|\|y\|,
\end{aligned}$$

we have $2\left\|\begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix}\right\|_{wh} \le \|u\|_h$.

To see the other inequality, given $\varepsilon > 0$. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix},$$

there exist x_1, x_2, y_1, y_2 such that $\begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ 0 & 0\end{bmatrix}\odot\begin{bmatrix} 0 & y_1 \\ 0 & y_2\end{bmatrix}$. Setting $x' = [x_1, x_2], y' = \begin{bmatrix} y_1 \\ y_2\end{bmatrix}$ with $\|x'\| = \|y'\|$, we rewrite $\begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix} = \begin{bmatrix} x' & 0 \\ 0 & 0\end{bmatrix}\odot\begin{bmatrix} 0 & y' \\ 0 & 0\end{bmatrix}$. Thus we may assume that $u = x'\odot y'$ with $\max\{\|x'x'^*\|, \|y'^*y'\|\} = \|x'\|\|y'\|$ and

$$2\left\|\begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix}\right\|_{wh} + \varepsilon > \left\|\begin{bmatrix} x' & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} x' & 0 \\ 0 & 0\end{bmatrix}^* + \begin{bmatrix} 0 & y' \\ 0 & 0\end{bmatrix}^*\begin{bmatrix} 0 & y' \\ 0 & 0\end{bmatrix}\right\|.$$

Hence we obtain $2\left\|\begin{bmatrix} 0 & u \\ 0 & 0\end{bmatrix}\right\|_{wh} \ge \|u\|_h$.

(b)

We let denote $X^\dagger = \{x^*\in\mathbb{B}(\mathcal{H}) \mid x\in X\}$ and also define a norm $\|\ \|_{wcb}$ on $X\otimes X^\dagger$ by

$$\|u\|_{wcb} = \inf\{\frac{1}{2}\|a\|\|x\|^2 \ \mid \ u = xa\odot x^* \in\mathbb{M}_n(X\otimes X^\dagger), x\in\mathbb{M}_{n,r}(X), a\in\mathbb{M}_r(\mathbb{C}), r\in\mathbb{N}\}$$

for $u\in\mathbb{M}_n(X\otimes X^\dagger)$ (c.f. [14], [6]).

It is easy to see that $\| \ \|_{wcb}$ also satisfies WI and WII. Since $\| \ \|_{wh}$ has another form [7] on $X \otimes X^\dagger$ as

$$\|u\|_{wh} = \inf\{w(a)\|x\|^2 \mid u = xa \odot x^* \in \mathbb{M}_n(X \otimes X^\dagger), x \in \mathbb{M}_{n,r}(X), a \in \mathbb{M}_r(\mathbb{C}), r \in \mathbb{N}\},$$

we have

$$\frac{1}{2}\|u\|_h \le \|u\|_{wcb} \le \|u\|_{wh} \le \|u\|_h, \quad u \in \mathbb{M}_n(X \otimes X^\dagger).$$

Thus it turns out from Corollary 3.3 that both $\| \ \|_{wh}$ and $\| \ \|_{wcb}$ are numerical radius norms affiliated with the operator space $X \otimes_h X^\dagger$ with the Haagerup norm $\| \ \|_h$.

We denote by $\mathcal{W}(X)$ the numerical radius operator space together with a numerical radius norm $\mathcal{W}$ affiliated with an operator space $(X, \mathcal{O}_n)$. We call $\mathcal{W}(X)$ a numerical radius operator space affiliated with $(X, \mathcal{O}_n)$. Let X, Y be operator spaces. It is clear that if $\varphi : X \longrightarrow Y$ is completely bounded, then $\varphi : \mathcal{W}_{(1)}(X) \longrightarrow \mathcal{W}_{(2)}(Y)$ is $\mathcal{W}$-completely bounded.

Lemma 3.7. *Let X, Y be operator spaces and $\mathcal{W}(X)$ a numerical radius operator space affiliated with X. If $\varphi : X \longrightarrow Y$ is completely bounded, then $\mathcal{W}(\varphi : \mathcal{W}(X) \longrightarrow \mathcal{W}_{min}(Y))_{cb} \le \mathcal{O}(\varphi)_{cb}$.*

Proof. It follows from

$$\begin{aligned} \mathcal{W}(\varphi)_{cb} &= \sup\{\frac{1}{2}\mathcal{O}(\varphi_n(x)) \mid \mathcal{W}(x) \le 1, x \in \mathbb{M}_n(X), n \in \mathbb{N}\} \\ &\le \sup\{\frac{1}{2}\mathcal{O}(\varphi_n(x)) \mid \frac{1}{2}\mathcal{O}(x) \le 1, x \in \mathbb{M}_n(X), n \in \mathbb{N}\} \\ &= \mathcal{O}(\varphi)_{cb}. \end{aligned}$$

□

Lemma 3.8. *Let X, Y be operator spaces and $\mathcal{W}(Y)$ a numerical radius operator space affiliated with Y. If $\varphi : X \longrightarrow Y$ is completely bounded, then $\mathcal{W}(\varphi : \mathcal{W}_{max}(X) \longrightarrow \mathcal{W}(Y))_{cb} \le \mathcal{O}(\varphi)_{cb}$.*

Proof. Assume that $\mathcal{O}(\varphi)_{cb} \le 1$. Let $x \in \mathbb{M}_n(\mathcal{W}_{\max}(X))$ with $\mathcal{W}_{\max}(x) \le 1$. Since $\mathcal{W}(Y)$ has a $\mathcal{W}$-complete isometry $\Phi : \mathcal{W}(Y) \longrightarrow w(\mathbb{B}(\mathcal{H}))$, we have $\mathcal{W}(\varphi_n(x)) = w(\Phi_n \circ \varphi_n(x))$. We note that $w(\Phi_n \circ \varphi_n(x)) \le \mathcal{W}_{\max}(x)$, since $\mathbb{M}_n(\mathcal{W}_{\max}(X/\ker(\Phi\circ\varphi))) \ni \tilde{x} \longmapsto \Phi_n\circ\varphi_n(x) \in \mathbb{M}_n(w(\mathbb{B}(\mathcal{H})))$ is isometric. Hence we have $\mathcal{W}(\varphi_n(x)) \le 1$. □

Lemma 3.9. *Let X, Y be operator spaces and $\mathcal{W}(X), \mathcal{W}(Y)$ numerical radius operator spaces affiliated with X, Y. If $\varphi : X \longrightarrow Y$ is completely bounded, then $\mathcal{O}(\varphi)_{cb} \le \mathcal{W}(\varphi : \mathcal{W}(X) \longrightarrow \mathcal{W}(Y))_{cb}$.*

Proof. It follows from

$$
\begin{aligned}
&\mathcal{O}(\varphi)_{cb}\\
&= \sup\{2\mathcal{W}\left(\begin{bmatrix} 0 & \varphi_n(x) \\ 0 & 0 \end{bmatrix}\right) \mid 2\mathcal{W}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \le 1, x \in \mathbb{M}_n(X), n \in \mathbb{N}\}\\
&\le \sup\{\mathcal{W}(\varphi_{2n}(y)) \mid \mathcal{W}(y) \le 1, y \in \mathbb{M}_{2n}(X), n \in \mathbb{N}\}\\
&= \mathcal{W}(\varphi)_{cb}.
\end{aligned}
$$

□

We let $\mathbb{O}$ denote the category of operator spaces, in which the objects are the operator spaces and the morphisms are the completely bounded maps. We also let $\mathbb{W}$ denote the category of numeical radius operator spaces with the morphisms being the $\mathcal{W}$-completely bounded maps. We have already obtained a functor $\mathcal{O} : \mathbb{W} \longrightarrow \mathbb{O}$ such that $\mathcal{O}(X) = 2\mathcal{W}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ symbolically. We have also found functors $\mathcal{W} : \mathbb{O} \longrightarrow \mathbb{W}$ which satisfy $\mathcal{O} \circ \mathcal{W}(X) = X$ for each operator space X. Combining the above Lemmas, $\mathcal{W}_{\max}$ and $\mathcal{W}_{\min}$ can be seen as the functors which embed $\mathbb{O}$ into $\mathbb{W}$ strictly.

Theorem 3.10. *Let X, Y be operator spaces. If $\varphi : X \longrightarrow Y$ is a linear map, then*

(1) $\mathcal{W}(\varphi : \mathcal{W}_{max}(X) \longrightarrow \mathcal{W}_{max}(Y))_{cb} = \mathcal{O}(\varphi : X \longrightarrow Y)_{cb}$,
(2) $\mathcal{W}(\varphi : \mathcal{W}_{min}(X) \longrightarrow \mathcal{W}_{min}(Y))_{cb} = \mathcal{O}(\varphi : X \longrightarrow Y)_{cb}$.

Proof. It is clear from Lemma 3.6, Lemma 3.7 and Lemma 3.8.

□

References

[1] T. Ando, *On the structure of operators with numerical radius one,* Acta Sci. Math. (Szeged), **34**(1973), 11–15.
[2] T. Ando and K. Okubo, *Induced norms of the Schur multiplier operator,* Linear Algebra Appl. **147**(1991), 181–199.
[3] E.G. Effros and Z-J. Ruan, *On the abstract characterization of operator spaces,* Proc. Amer. Math. Soc. **119** (1993), 579–584.
[4] E.G. Effros and Z-J. Ruan, *Operator Spaces,* London Math. Soc. Mono. New Series, Vol. 23, Oxford Univ. Press, 2000.
[5] U. Haagerup and P. de la Harpe, *The numerical radius of a nilpotent operator on a Hilbert space,* Proc. Amer. Math. Soc. **115** (1992), 371–379.
[6] T. Itoh and M. Nagisa, *Numerical Radius Norm for Bounded Module Maps and Schur Multipliers*, to appear in Acta Sci. Math. (Szeged).
[7] T. Itoh and M. Nagisa, *The numerical radius Haagerup norm and Hilbert space square factorizations*, preprint.

[8] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Adv. Math. 78, Cambridge Univ. Press, 2002.

[9] V. I. Paulsen and C. Y. Suen, *Commutant representations of completely bounded maps,* J. Operator Theory **13**(1985), 87–101.

[10] G. Pisier, *Introduction to operator space theory*, London Math.Soc. Lecture Note Series 294, Cambridge Univ. Press, 2003.

[11] Z-J. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal. **76**, (1988), 217–230.

[12] R. R. Smith, *Completely bounded maps between C^*-algebras,* J. London Math. Soc. **27**, (1983), 157–166.

[13] C.-Y. Suen, *The numerical radius of a completely bounded map,* Acta Math. Hungar. **59**, (1992), 283–289.

[14] C.-Y. Suen, *Induced completely bounded norms and inflated Schur product,* Acta Sci. Math. (Szeged) **66**, (2000), 273–286.

[15] C.-Y. Suen, W_ρ *completely bounded maps,* Acta Sci. Math. (Szeged) **67**, (2001), 747–760.

*Department of Mathematics, Gunma University, Gunma 371-8510, Japan

E-mail address: `itoh@edu.gunma-u.ac.jp`

**Department of Mathematics and Informatics, Chiba University, Chiba 263-8522, Japan

E-mail address: `nagisa@math.s.chiba-u.ac.jp`